Problem Set 7 due April 22, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider the matrix $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ for any numbers a, b, c.

(1) Give a condition, in terms of a, b, c, for the matrix A to be invertible. *(6 points)*

(2) Assuming a, b, c satisfy the condition you found in the previous part, use the cofactor formula for the inverse to compute A^{-1}. *(10 points)*

(3) Use Cramer's rule to compute the solution to the equation:

$$A\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for any numbers x, y, z. *(9 points)*

Solution: (1) The matrix A is invertible if and only if its determinant is non-zero. You may compute the determinant by any method you wish (e.g. cofactor expansion) and you should get:

$$\det A = a^3 + b^3 + c^3 - 3abc$$

Grading Rubric

- Gave condition $\det A \neq 0$ and worked out a correct formula for the determinant *(6 points)*
- Gave condition $\det A \neq 0$, and gave without proof a correct formula for $\det A$ *(4 points)*
- Gave condition $\det A \neq 0$, and gave without proof a correct formula for $\det A$ *(2 points)*
- No correct answer *(0 points)*

(2) The cofactor formula for the inverse is:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{21} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

All the cofactors must be computed according to their definition (i.e. the students must explain how to apply the formula for the cofactors, and not just state the answer directly) and the answer should be:

$$A^{-1} = \frac{1}{a^3+b^3+c^3-3abc}\begin{bmatrix} a^2-bc & c^2-ab & b^2-ac \\ b^2-ac & a^2-bc & c^2-ab \\ c^2-ab & b^2-ac & a^2-bc \end{bmatrix}$$

Grading Rubric

- Correct answer with explanation of all the steps *(10 points)*
- Correct answer with explanation of all the steps, but minor computational errors *(8-9 points)*
- Correct answer, but no explanation of how cofactors are computed *(6 points)*
- Incorrect answer or method *(0 points)*

(3) Cramer's rule says that the solution to the system of equations is:

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

where:

$$\boldsymbol{v}_1 = \frac{1}{\det A}\det\begin{bmatrix} x & b & c \\ y & a & b \\ z & c & a \end{bmatrix}$$
$$\boldsymbol{v}_2 = \frac{1}{\det A}\det\begin{bmatrix} a & x & c \\ c & y & b \\ b & z & a \end{bmatrix}$$
$$\boldsymbol{v}_3 = \frac{1}{\det A}\det\begin{bmatrix} a & b & x \\ c & a & y \\ b & c & z \end{bmatrix}$$

One computes the determinants above by cofactor expansion (say along the column which has x, y, z) and gets the answer:

$$\boldsymbol{v}_1 = \frac{x(a^2-bc)+y(c^2-ab)+z(b^2-ac)}{a^3+b^3+c^3-3abc}$$
$$\boldsymbol{v}_2 = \frac{x(b^2-ca)+y(a^2-bc)+z(c^2-ab)}{a^3+b^3+c^3-3abc}$$
$$\boldsymbol{v}_3 = \frac{x(c^2-ab)+y(b^2-ca)+z(a^2-bc)}{a^3+b^3+c^3-3abc}$$

Grading Rubric

- Correct answer using Cramer's rule, with sufficient explanations *(9 points)*
- Correct answer using Cramer's rule, with sufficient explanations, but minor errors *(7-8 points)*

- Correct answer, but used a different method than Cramer's rule (e.g. the explicit formula for the inverse) *(6 points)*
- Correct answer, but insufficient explanation *(3 points)*
- Missing or very incorrect answer *(0 points)*

Problem 2:

Let $\boldsymbol{v} \in \mathbb{R}^n$ be a vector of length 1, and consider the matrix $A = I - 2\boldsymbol{v}\boldsymbol{v}^T$.

(1) Show that $\boldsymbol{v}$ is an eigenvector of A. What is the corresponding eigenvalue? *(5 points)*

(2) Describe $n-1$ other eigenvectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{n-1}$ of A, which together with $\boldsymbol{v}$ form a basis of $\mathbb{R}^n$. What are the corresponding eigenvalues of these $n-1$ eigenvectors? *(10 points)*

(3) What is the rank of the matrix A? How do you know? *(5 points)*

Solution: (1) We have:

$$A\boldsymbol{v} = (I - 2\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{v} = \boldsymbol{v} - 2\boldsymbol{v}\underbrace{\boldsymbol{v}^T\boldsymbol{v}}_{=1} = -\boldsymbol{v}$$

so $\boldsymbol{v}$ is an eigenvector corresponding to the eigenvalue -1.

Grading Rubric

- Correct answer with justification *(5 points)*
- Correct answer without justification *(3 points)*
- Incorrect answer (bar minor typos) *(0 points)*

(2) As we have seen in Problem 1 on the previous Problem Set, any vector $\boldsymbol{w}$ which is perpendicular to $\boldsymbol{v}$ is preserved by the matrix A, i.e.:

$$A\boldsymbol{w} = (I - 2\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{w} = \boldsymbol{w} - 2\boldsymbol{v}\underbrace{\boldsymbol{v}^T\boldsymbol{w}}_{=0} = \boldsymbol{w}$$

So any vector $\boldsymbol{w} \perp \boldsymbol{v}$ is an eigenvector of A corresponding to the eigenvalue 1. Since the orthogonal complement $v^\perp$ is a subspace of dimension $n-1$, we let $\boldsymbol{v}_1, ..., \boldsymbol{v}_{n-1}$ be any basis of $v^\perp$, and they will all be eigenvectors corresponding to the eigenvalue 1.

Grading Rubric

- Correct answer with justification *(10 points)*
- Correct answer without justification (e.g. without saying why $A\boldsymbol{w} = \boldsymbol{w}$ for $\boldsymbol{w} \perp \boldsymbol{v}$) *(5 points)*

- Incorrect answer *(0 points)*

(3) The rank of A is n, because all the eigenvalues of A are non-zero (which implies $\det A \neq 0$, which is equivalent to having full rank for a square matrix).

Grading Rubric

- Correct answer with justification (not necessarily our eigenvalues argument) *(5 points)*
- Correct answer without justification *(2 points)*
- Incorrect answer *(0 points)*

Problem 3:

Consider the matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

(1) Prove that $\text{Tr}(AB) = \text{Tr}(BA)$ by computing them explicitly. *(10 points)*

(2) Use the previous part to prove that $\text{Tr}(D) = \text{Tr}(VDV^{-1})$ for any 2×2 square matrix D and any invertible 2×2 matrix V. *(5 points)*

Solution: (1) Explicitly:

$$AB = \begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix} \qquad \Rightarrow \qquad \text{Tr}(AB) = ax + bz + cy + dt$$

$$BA = \begin{bmatrix} xa + yc & xb + yd \\ za + tc & zb + td \end{bmatrix} \qquad \Rightarrow \qquad \text{Tr}(BA) = xa + zb + yc + td$$

The right-hand sides are equal to each other.

Grading Rubric

- Correct computation *(10 points)*
- Minor errors *(7-8 points)*
- Incorrect computation or no answer *(0 points)*

(2) Just apply the identity $\text{Tr}(AB) = \text{Tr}(BA)$ for $A = DV^{-1}$ and $B = V$.

Grading Rubric

- Correct proof using part (1) *(5 points)*

- Correct proof using a different method *(3 points)*
- Incorrect or no proof *(0 points)*

Problem 4:

Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Prove that the any non-zero eigenvalue of the square matrix AB is also an eigenvalue of the matrix BA. *(10 points)*

Solution: Assume $\lambda \neq 0$ is an eigenvalue of AB. This means that:

$$AB\boldsymbol{v} = \lambda \boldsymbol{v} \tag{1}$$

for some non-zero vector $\boldsymbol{v}$. Multiplying the relation above on the left with B gives:

$$BAB\boldsymbol{v} = \lambda B\boldsymbol{v}$$

If we let $\boldsymbol{w} = B\boldsymbol{v}$, then we get:

$$BA\boldsymbol{w} = \lambda \boldsymbol{w}$$

This implies that λ is an eigenvalue of BA, because $\boldsymbol{w} \neq 0$ (otherwise, the fact that $\boldsymbol{w} = B\boldsymbol{v} = 0$ would force $\lambda = 0$ in (1), which the problem assumes is not the case).

Grading Rubric

- Correct argument *(10 points)*
- Partially correct argument (i.e. which could be made correct using a little work) *(5 points)*
- Incorrect or no argument *(0 points)*

Problem 5:

Consider the matrix:

$$A = \begin{bmatrix} -3 & 4 & -8 \\ 3 & -5 & 8 \\ 2 & -3 & 5 \end{bmatrix}$$

(1) Find an eigenvalue λ of A, and compute an eigenvector $\boldsymbol{v}$. *(10 points)*

(2) Compute a vector $\boldsymbol{w}$ such that $(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}$ and a vector $\boldsymbol{z}$ such that $(A - \lambda I)\boldsymbol{z} = \boldsymbol{w}$. *(10 points)*

(3) Consider the matrix $V = \left[\ \boldsymbol{v} \mid \boldsymbol{w} \mid \boldsymbol{z}\ \right]$ and compute:

$$V^{-1}AV$$

Congratulations: you just computed the Jordan normal form of A. *(10 points)*

Solution: (1) Let us compute the characteristic polynomial of A:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & 4 & -8 \\ 3 & -5-\lambda & 8 \\ 2 & -3 & 5-\lambda \end{bmatrix} =$$

$$= (-3-\lambda) \cdot \det \begin{bmatrix} -5-\lambda & 8 \\ -3 & 5-\lambda \end{bmatrix} - 4 \cdot \det \begin{bmatrix} 3 & 8 \\ 2 & 5-\lambda \end{bmatrix} + (-8) \cdot \det \begin{bmatrix} 3 & -5-\lambda \\ 2 & -3 \end{bmatrix} =$$

$$= (-3-\lambda)((-5-\lambda)(5-\lambda)+24) - 4(3(5-\lambda)-16) - 8(-9+2(5+\lambda)) = -\lambda^3 - 3\lambda^2 - 3\lambda - 1 = -(\lambda+1)^3$$

So the unique eigenvalue is $\lambda = -1$, with algebraic multiplicity 3. By definition, an eigenvector $\boldsymbol{v}$ corresponding to this eigenvalue is an element in the following nullspace:

$$\boldsymbol{v} \in N(A+I) = N\left(\begin{bmatrix} -2 & 4 & -8 \\ 3 & -4 & 8 \\ 2 & -3 & 6 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

(the procedure above is Gaussian or Gauss-Jordan elimination, and we expect students to show all the steps) so:

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Grading Rubric: 5 points for the eigenvalue and 5 points for the eigenvector (for each of them, we will take off 1-2 points for computational errors, and take off 3 points for insufficient explanations).

(2) Let's solve for $\boldsymbol{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying the identity:

$$\begin{bmatrix} -2 & 4 & -8 \\ 3 & -4 & 8 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

As before, you do this by Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & 4 & -8 & 0 \\ 3 & -4 & 8 & 2 \\ 2 & -3 & 6 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the system is equivalent to:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Let's solve for $\boldsymbol{z} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ satisfying the identity:

$$\begin{bmatrix} -2 & 4 & -8 \\ 3 & -4 & 8 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

As before, you do this by Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & 4 & -8 & 2 \\ 3 & -4 & 8 & 1 \\ 2 & -3 & 6 & 0 \end{array}\right] \quad \rightsquigarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the system is equivalent to:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{z} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Grading Rubric: 5 points for each of $\boldsymbol{w}$ and $\boldsymbol{z}$, of which:

- Correct computation using row reduction *(5 points)*
- Correct computation using row reduction, but minor computational error *(4 points)*
- Correct computation, but not methodical (e.g. back substitution) *(3 points)*
- Incorrect computation or method *(0 points)*

(3) With the vectors we computed in the previous parts, we set:

$$V = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

The determinant of V is 1, which one can see by doing cofactor expansion on the last row. Similarly, we can compute the inverse of V by cofactor expansion, and we get:

$$V^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -3 & 6 \\ -1 & 2 & -4 \end{bmatrix}$$

Then we can explicitly compute:

$$V^{-1}AV = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \Leftrightarrow \quad A = V \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} V^{-1}$$

This shows that $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ is the Jordan normal form of A.

Grading Rubric: 5 points for the inverse (-2 points if it was not computed using a specific method, such as using cofactor expansion or Gauss-Jordan elimination) and 5 points for the correct computation of $V^{-1}AV$.